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The Cauchy Problem for Elastic Waves in an Anisotropic Medium

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THE CAUCHY PROBLEM FOR ELASTIC WAVES IN AN ANISOTROPIC MEDIUM

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CONTENTS

The propagation of elastic waves in a homogeneous solid is governed by a hyperbolic system of three linear second-order partial differential equations with constant coefficients. When the solid is also isotropic, the form of these equations is well known and provides the foundation of the conventional theory of elasticity (Love 1944).

The explicit solution of the initial value, or Cauchy, problem for the isotropic case was found by Poisson, and in a different way by Stokes (1883). If the initial disturbance is sharp and concentrated, the resulting disturbance at a field point will consist of an initial sharp pressure wave, a continuous wave for a certain period, and a final sharp shear wave. The disturbance then ceases.

Here we shall consider a medium which is homogeneous but not isotropic, and will describe, using Fourier transforms, the elastic waves produced by a local initial disturbance. The solution again consists of a continuous wave which lasts for a definite period of time, and a number of sharp waves, but the detailed nature of the waves may, in highly anisotropic media, be very different and much more complicated. The continuous wave may arrive at a field point in advance of the first sharp wave, though it will always terminate with the last sharp wave. The number of the sharp waves may not exceed 75. The solution appears as the sum of three modes, which correspond to the three sheets of a certain wave surface. The geometry of this surface, which may be quite complicated (Musgrave 1954*a*), qualitatively determines the nature of the solution.

These calculations may serve as a foundation for the study of time-dependent elastic waves. There is also mathematical interest in this example of a hyperbolic system for which the wave surface may have certain types of singularities not usually considered in the existing general theory of hyperbolic differential equations.

1. The basic equations

An anisotropic solid medium is described by 21 elastic constants which form a Cartesian tensor of the fourth order (Wooster 1938, p. 234). We shall here denote by c_{pqrs} the usual elastic constants divided by density ρ, and we note the symmetry conditions

$$c_{pqrs} = c_{rspq} = c_{qprs} \quad (p, q, r, s = 1, 2, 3). \tag{1·1}$$

In the absence of body force the equations of motion are (Love 1944)

$$\frac{\partial^2 u_p}{\partial t^2} = \frac{\partial^2}{\partial x^q \partial x^s}(c_{pqrs} u_r), \tag{1·2}$$

where the dependent variables u_p $(p = 1, 2, 3)$ are Cartesian components of the elastic displacement vector, the x_q are Cartesian co-ordinates in space, and t is time. Summation over q, r and s in (1·2) is understood according to the rules of Cartesian tensor calculus.

It is assumed that the constants c_{pqrs} have numerical values such that the strain energy W, given by

$$2W = \rho c_{pqrs} \epsilon_{pq} \epsilon_{rs} \tag{1·3}$$

is positive definite, for symmetric stress components $\epsilon_{pq} = \epsilon_{qp}$.

We seek solutions of (1·2) for the initial conditions of Cauchy's problem:

$$u_p = f_p(\mathbf{x}), \quad \partial u_p/\partial t = g_p(\mathbf{x}) \quad \text{for} \quad t = 0. \tag{1·4}$$

However, we shall restrict our study to the case where only the initial velocities $g_p(\mathbf{x})$ are different from zero. Indeed, if the solution of (1·2) with $u_p = 0$, $\partial u_p/\partial t = g_p(\mathbf{x})$ for $t = 0$ is denoted by $u_p[g(\mathbf{x}), t]$, then the solution of (1·2) (supposing body force per unit density terms $h_p(\mathbf{x}, t)$ inserted on the right), and with initial conditions (1·4), is given by

$$u_p(\mathbf{x}, t) = \frac{\partial}{\partial t} u_p[f(\mathbf{x}), t] + u_p[g(\mathbf{x}), t] + \int_0^t u_p[h(\mathbf{x}, \tau), t-\tau]\, d\tau. \tag{1·5}$$

This formula is an instance of Stokes's and Duhamel's rules (Stokes 1883, p. 263; Courant & Hilbert 1937, p. 165).

The Fourier transform notation to be used is

$$Ff(\mathbf{x}) = \frac{1}{(2\pi)^{\frac{3}{2}}} \int f(\mathbf{y})\, e^{i\mathbf{x}.\mathbf{y}}\, d\mathbf{y} \tag{1·6}$$

with

$$F^{-1}f(\mathbf{x}) = \frac{1}{(2\pi)^{\frac{3}{2}}} \int f(\mathbf{y})\, e^{-i\mathbf{x}.\mathbf{y}}\, d\mathbf{y}.$$

Here $FF^{-1} = F^{-1}F = 1$, and the integrations are taken over the entire y-space with $d\mathbf{y} = dy_1 dy_2 dy_3$. The scalar product has been written $\mathbf{x}.\mathbf{y}$.

Applying the transform, we shall take as initial conditions

$$Fu_p = 0, \quad \frac{\partial}{\partial t} Fu_p = Fg_p(\mathbf{x}) \quad (t = 0). \tag{1·7}$$

At a later stage we shall use Dirichlet's theorem for Fourier series (Courant & Hilbert 1931, p. 66) in the form

$$\lim_{R\to\infty} \int_a^b f(\xi) \frac{\sin R\xi}{\xi}\, d\xi = \begin{cases} 0 & (0 < a < b) \\ \pi f(0) & (a < 0 < b) \end{cases} \tag{1·8}$$

for continuous functions $f(\xi)$.

2. The slowness and wave surfaces

A surface $S: \phi(x, t) = 0$ in space-time is said to be characteristic with respect to the system (1·2) if

$$\det\left[\left(\frac{\partial \phi}{\partial t}\right)^2 \delta_{pr} - c_{pqrs} \frac{\partial \phi}{\partial x^q} \frac{\partial \phi}{\partial x^s}\right] = 0. \tag{2·1}$$

The characteristic or wave cone W with vertex at the origin of space-time is the envelope of the characteristic planes which pass through the origin. As we shall see in § 3, the importance of the wave cone for the elastic waves is that its convex envelope limits the region of influence of a disturbance originating at the origin.

This wave cone has in general three sheets, and is constructed as follows.

First we form the slowness, or 'normal' cone S with vertex at the origin and equation

$$S(t,y) = \det\,[t^2\delta_{pr} - c_{pqrs}y_q y_s] = 0. \tag{2·2}$$

For given y_q there are, in general, three values of t^2 for which this relation is satisfied. We shall denote them by

$$t^2 = v_N^2(y) \quad (N = 1, 2, 3),$$

where the 'velocity' functions $v_N(y)$ thus defined are labelled so that

$$v_1^2(y) \geqslant v_2^2(y) \geqslant v_3^2(y). \tag{2·3}$$

The $v_N(y)$ are positively homogeneous of the first degree in y: thus $v_N(ky) = |k|\,v_N(y)$. Clearly the $v_N(y)$ are the characteristic roots of the matrix

$$K_{pr}(y) = c_{pqrs}y_q y_s.$$

We will show that this matrix is positive definite, and hence that its three characteristic roots $v_N^2(y)$ are all positive. This follows from our assumption (1·3) that the strain energy is positive definite. Setting $\epsilon_{pq} = x_p y_q + y_p x_q$ in (1·3), and rearranging with the help of (1·1), we find

$$c_{pqrs}x_p x_r y_q y_s > 0$$

unless all x_r or all y_s are zero, and this proves that $K_{pr}(y)$ is positive definite as required.

We now make precise the notion of a sheet. A sheet of S is the locus defined by

$$t^2 = v_N^2(y),$$

where N is *one* of the values 1, 2, 3. Thus S consists of three sheets, S_1, S_2, S_3. If two of the $v_N(y)$ coincide, then the corresponding sheets meet. Because of our convention that

$$v_1^2(y) \geqslant v_2^2(y) \geqslant v_3^2(y),$$

however, the sheets of S do not pass through each other.

We may distinguish two types of slowness cone as follows. If the equality signs in (2·3) never hold, that is, if the sheets of S meet only at the origin, then we shall call S *regular*. Leray (1953) uses the term 'regularly hyperbolic' to describe an equation with slowness cone having this property. On the other hand, if the sheets of S meet along one or more generators of the slowness cone, we shall call S *singular*. This singular case requires a special discussion which is given in § 6 below.

The reason for the term 'slowness cone' may be stated briefly. If a plane wave solution

$$u_p = A_p f(y_r x_r - t)$$

satisfies (1·2), then y_r must satisfy

$$\det\,[\delta_{pr} - c_{pqrs}y_q y_s] = 0,$$

which, if we write $y_r = \eta_r\,|y|$ with $|\eta| = 1$, takes the form

$$\det\left[\frac{1}{|y|^2}\delta_{pr} - c_{pqrs}\eta_q\eta_s\right] = 0.$$

Since the wave velocity is $v = 1/|y|$ we find

$$\det[v^2\delta_{pr} - c_{pqrs}\eta_q\eta_s] = 0. \tag{2·4}$$

A comparison with (2·2) shows that the radius vector $|y|$ of (2·2) for $t = 1$ is the slowness $1/v(y)$ for that particular direction. Fuller detail regarding plane wave solutions may be found in Synge (1956, 1957).

The section of the slowness cone $S = S(t, y)$ by the hyperplane $t = 1$ gives the slowness surface $S = S(1, y)$. This two-dimensional surface in the 3-space of the variables y_r is algebraic of the sixth degree, and bounded. In the regular case it consists of three concentric and non-intersecting sheets, each of which is a simple closed surface. In the singular case the three surfaces have common points, which are said to be multiple points of the slowness surface $S(1, y)$.

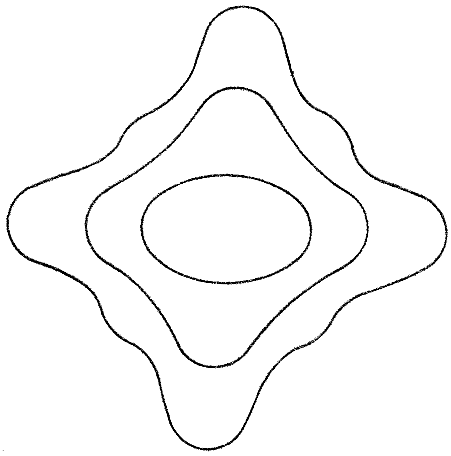

FIGURE 1. Sketch of a central plane section of a slowness surface. No multiple points are present, so the system of differential equations is regularly hyperbolic.

To construct the wave cone, we note that the normal to the sheet S_N $(N = 1, 2, 3)$ of the slowness cone has direction numbers $(\partial v_N/\partial y_r, -1)$. We reflect these normals in the plane $t = 0$, and get $(\partial v_N/\partial y_r, +1)$. As y ranges over S_N, with τ fixed, these reflected normals generate in space-time a surface which is the sheet W_N of the wave cone W. The equation of this sheet is to be found by eliminating from the equations

$$\frac{x_r}{t} = \frac{\partial v_N(\mathbf{y})}{\partial y_r}, \tag{2·5}$$

the ratios of the three variables y_r. Since $v_N(\mathbf{y})$ is positively homogeneous of degree one, its first derivatives have degree zero and only the two ratios of the y_r enter. For $t =$ const. the cross-section of the wave cone is a three-sheeted closed surface in physical space—the wave surface $W(t)$. When the vertex of the wave cone is translated to the point x, the wave surface will be denoted by $W(\mathbf{x}, t)$ and its sheets by $W_N(\mathbf{x}, t)$ (figure 2).

As the qualitative properties of the elastic waves are largely determined by the configuration of the wave surface, we first study it from a geometrical viewpoint. It is sufficient to consider $t = 1$. From (2·3) it follows that $W(0, 1)$ is the polar reciprocal with respect to the unit sphere of the slowness surface S (Salmon 1882, p. 580). The radius vector $\mathbf{y}$ of a point of S has the direction of the normal to W at the corresponding point of W, and vice versa.

Setting $t = 1$ we find for the general equation of S:

$$S(\mathbf{y}) = \det[\delta_{pr} - c_{pqrs} y_q y_s] = 0,$$

which is of the sixth degree, and, as we have seen, S consists of three real sheets. In the isotropic case these are concentric spheres, and the two outermost coincide. In general each separate sheet S_N of S is a piecewise analytic simple closed surface, having no multiple points, since, if a multiple point existed, the line joining it to the origin would meet the three sheets in more than six points, which is impossible. However, two, or even all three, sheets may meet, so that the slowness surface as a whole may have double or triple points.

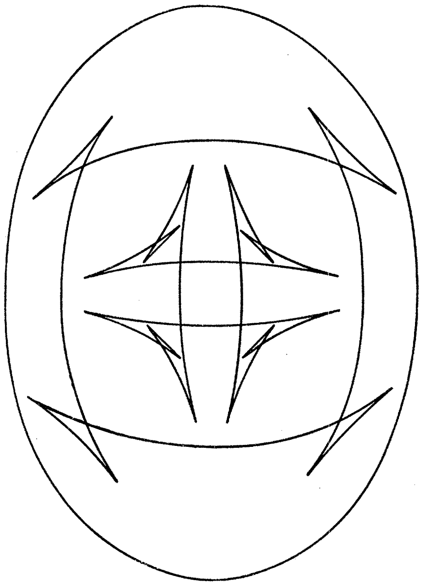

FIGURE 2. Sketch of the central section of the wave surface dual to figure 1. The point singularities of the wave surface are dual to the tangential singularities of the slowness surface. The class number shown in the figure is 18; the class of the entire wave surface may be greater.

However, if the innermost sheet S_1 of S does not meet the other two, then this sheet is convex. Indeed any straight line can meet S_1 in at most two points, since it must meet S_2 and S_3 in at least two points each.

The three-sheeted wave surface W is also an algebraic surface, of degree equal to the class of S, that is, equal to the greatest number of tangent planes to S which pass through a given straight line. The class number of a surface S of degree $n = 6$ may not exceed $n(n-1)^2 = 150$ (Salmon 1882, p. 248), and will be less if S has point singularities of certain kinds. A complete analysis of these singularities and so of their effect upon the class number of S is far too complicated to be undertaken here. We shall later discuss the class number for two particular anisotropic types of crystal—the cubic and hexagonal. As W is symmetric with respect to the origin, any half line from the origin will meet W at most 75 times. A set of parametric equations for W has been found by Musgrave (1954*a*), while extensive numerical studies of hexagonal and cubic media are given by Musgrave (1954*b*) and Miller & Musgrave (1956).

The singularities of W are the duals of those of S. To a double point of S corresponds a double tangent of W, and vice versa. Since S has no cusps or cuspidal edges, W can have no lines of inflexion. However, S may have lines of inflexion, and so W will then possess cuspidal edges.

We shall need to study the leading wave front of a wave originating from a point source. Since polar reciprocation reverses the inclusion relation for surfaces symmetric about the origin, the leading wave front is in general the reciprocal of the innermost sheet S_1 of the slowness surface.

Any straight line which meets S_1 must also meet each of the other two sheets of S in two points, for a total of four. Since S has degree six, it follows that no straight line can meet S_1 in more than two points. This implies that S_1 is a strictly convex surface, for the straight line joining any two points of S_1 has one and only one segment lying within S_1.

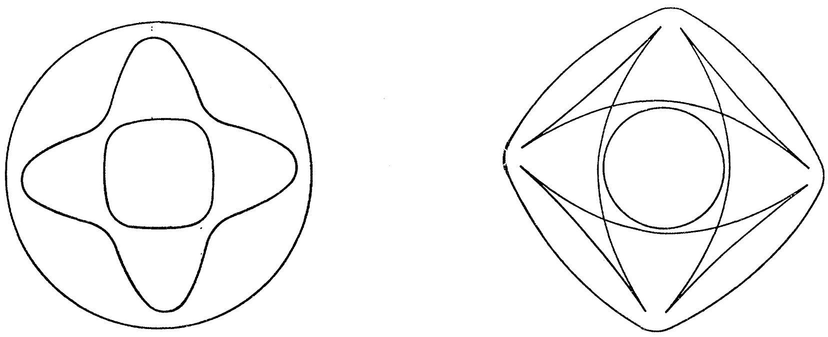

FIGURE 3. Co-ordinate-plane sections of the slowness and wave surfaces, respectively, in the cubic case with $b > 0$. The number of sharp waves will be 5, except in directions sufficiently close to the co-ordinate axes, where it is 3.

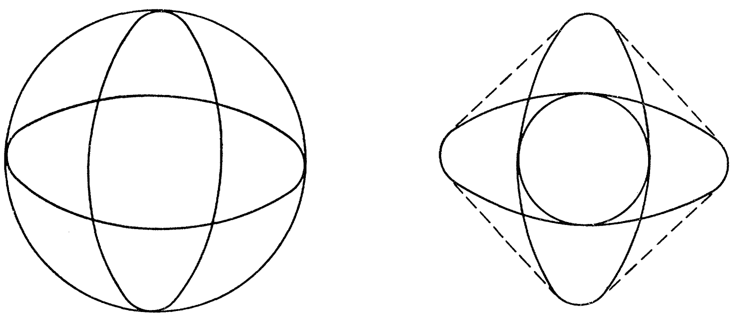

FIGURE 4. Co-ordinate-plane sections of the slowness and wave surfaces, respectively, in the cubic case with $b = 0$. The four double points on the inner sheets of S correspond to the four dashed lines which form the convex completion of W. Comparison with figure 3 shows that these lines, which are generators of a ruled surface portion of W, arise from coalescence of certain parts of the cusps and of the convex outer sheet.

When the innermost sheet S_1 does not meet the other two sheets of S, it has a continuously turning tangent plane. Then the dual surface W is also strictly convex, and forms the outermost of the three sheets of the wave surface W.

However, if S_1 meets the other sheets of S in certain multiple curves or points, which correspond to equal values for two or more of the roots $v_N(\mathbf{y})$, the construction of $\overline{W}$ is more complicated. At such multiple points the inner surface S_1, which has the equation $\bar{v}_1(\mathbf{y}) = 1$ (with $\bar{v}_1(\mathbf{y})$ the largest of the three roots $v_N(\mathbf{y})$), may not have a continuously turning tangent plane. Such a discontinuity of the normal to S_1 corresponds to a discontinuity of position, but not of the normal, in $\overline{W}$. As the tangent plane to S_1 turns about the given double point, the dual point of $\overline{W}$ moves along a given double straight line. To double curves on S_1 there will correspond portions of ruled surfaces of $\overline{W}$, while conical points of S_1 go over into planar regions of $\overline{W}$, as for example in the cubic case $b = 0$ (figures 3, 4), which is discussed in § 7 (*b*). In fact $\overline{W}$ is the boundary of the smallest convex region containing all three sheets of W, and so is called the convex envelope of W.

3. Sharp and continuous waves, dependence domains, and lacunae

The solution of our initial value problem will be expressed as the sum of a surface integral (5·26) and a volume integral (5·38). The surface integral, which we shall call the sharp wave, is taken over the wave surface $W(\mathbf{x}, t)$ with centre x, the field point. The volume integral, to be known as the continuous wave, is taken over the interior of the convex envelope $\overline{W}(\mathbf{x}, t)$, excluding the innermost region $R_0(\mathbf{x})$, which will be defined later.

The qualitative nature of the solution will now be examined from two complementary points of view. Let us first consider a source, or initial disturbance, concentrated in a neighbourhood of x having diameter δ. If we draw in space-time the direct, or forward, wave cone W with vertex at $(\mathbf{x}, 0)$, and hence with cross-section $W(\mathbf{x}, t)$ at time t, then the sharp wave generated by our disturbance will occupy a neighbourhood of thickness δ of the wave cone. Thus the spatial region of influence of the sharp wave is a number of shells, of thickness δ, enclosing the three sheets of $W(\mathbf{x}, t)$. The volume of these shells is approximately proportional to t^2. In contrast, the continuous wave emanating from this source occupies a region which is a neighbourhood of the interior of the convex closure $\overline{W}(\mathbf{x}, t)$, and has volume nearly proportional to t^3.

The complementary point of view is that of the observer at a field point $\mathbf{x}$ at time t, and of the domain of dependence of the solution $u_p(\mathbf{x}, t)$ upon the initial data. Again, let the initial disturbance be concentrated in a region R of diameter δ, which contains the origin. Since the differential equations are linear there is no loss of generality in this assumption. As the time t increases the observer's wave surface $W(\mathbf{x}, t)$ with centre x will expand. Since the sharp wave is a surface integral over $W(\mathbf{x}, t)$, the first sharp wave will arrive at $\mathbf{x}$ at the instant the outer sheet $W(\mathbf{x}, t)$ encounters the region R. After a time interval $\delta v(\mathbf{x})$, where $v(\mathbf{x})$ is the phase velocity associated with that particular sheet, $W(\mathbf{x}, t)$ will pass over R, and the first sharp wave will terminate. Subsequent sharp waves will reach $\mathbf{x}$ as the remaining sheets of $W(\mathbf{x}, t)$ arrive at the region R, and each one will endure for a time of order δ. Thus for small t the various sharp waves travelling in a given direction may overlap, but after a sufficiently long time they become separated.

The continuous wave reaches the field point $\mathbf{x}$ at the instant the convex envelope $\overline{W}(\mathbf{x}, t)$ touches R. This may be before the first sharp wave, unless $W(\mathbf{x}, t)$ is itself convex. Then the continuous wave will endure until the last sharp wave has passed, an interval of time proportional to the distance to the origin. The disturbance at $\mathbf{x}$ thereafter ceases.

Thus the terms 'sharp' and 'continuous' describe, in contrasting fashion, the profiles of the two types of wave originating from a concentrated initial disturbance (figure 5).

The domain of dependence may be visualized in space-time by drawing the backward or retrograde wave cone with vertex at the observer's point $(\mathbf{x}, t)$, and noting the intersection of the sheets of that cone with the initial hyperplane $t = 0$. This intersection is exactly congruent to $W(\mathbf{x}, t)$. The domain of dependence of the sharp wave consists of the intersection $W(\mathbf{x}, t)$ of the sheets themselves, a locus of dimension 2. The domain of dependence for the continuous wave is the 3-dimensional region which is interior to the convex envelope $\overline{W}(\mathbf{x}, t)$ of the wave surface, except, as noted before, for the central part of that wave surface.

That the domain of dependence is bounded by the convex envelope $\overline{W}(\mathbf{x}, t)$ follows from the general theory of hyperbolic differential equations, at least in the regularly hyperbolic case when the slowness surface has no multiple points (Leray 1953, pp. 128, 168). For completeness we shall give a direct proof that $u_p(\mathbf{x}, t)$ depends only on the initial data in and on the convex envelope $\overline{W}(\mathbf{x}, t)$. This will follow if it can be shown that the vanishing

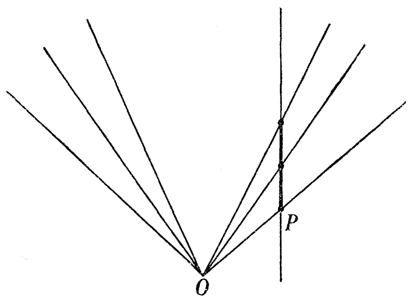

FIGURE 5. Section in space (horizontally) and time (vertically) of an impulse, originating at O, which meets the 'world line' of a point P fixed in space. The sharp waves are carried on the sheets of the wave cone, while the continuous wave occupies the region enclosed between the inner and outer wave sheets.

of the data in $\overline{W}(\mathbf{x}, t)$ ensures that $u_p(\mathbf{x}, t)$ shall be zero. To show this let the convex retrograde wave cone with vertex $(\mathbf{x}, t)$ in space-time be drawn, meeting the initial surface $t = 0$ in $\overline{W}(\mathbf{x}, t)$. Then this wave cone meets the plane $t = h$ in the convex surface $\overline{W}(\mathbf{x}, t-h)$. The normal to the wave cone has direction numbers $(y_r, 1)$ which according to the reciprocal polar relation are co-ordinates of a point on the slowness cone S. Since we consider the convex envelope of the wave cone, the y_r define a point of the innermost sheet S_1.

Let us denote time derivatives by a dot, and derivatives with respect to x^r by a subscript r following the index of the component: thus $\dot{u}_p$, $u_{p,r}$. We consider the total of kinetic and strain energy in the spatial region $\overline{W}(\mathbf{x}, t-h)$ at time h, and show that this energy cannot increase.

Thus let
$$E(h) = \frac{1}{2}\int (\dot{u}_p \dot{u}_p + c_{pqrs} u_{p,q} u_{r,s})\, \mathrm{d}V \tag{3·1}$$

denote the positive definite energy integral (see (1·3)), taken over the interior of $\overline{W}(\mathrm{x}, t-h)$. If, as h increases, σ denotes the inward normal velocity of the boundary of this region, then

$$\frac{\mathrm{d}E(h)}{\mathrm{d}h} = \int (\dot{u}_p \ddot{u}_p + c_{pqrs} \dot{u}_{p,q} u_{r,s})\, \mathrm{d}V$$
$$-\frac{1}{2}\int \sigma(\dot{u}_p \dot{u}_p + c_{pqrs} u_{p,q} u_{r,s})\, \mathrm{d}\Sigma.$$

From the equations of motion (1·2), the first integral will become

$$\int (\dot{u}_p c_{pqrs} u_{r,sq} + c_{pqrs} \dot{u}_{p,q} u_{r,s})\,\mathrm{d}V$$
$$= \int (\dot{u}_p c_{pqrs} u_{r,s})_{,q}\,\mathrm{d}V$$
$$= \int c_{pqrs} \dot{u}_p u_{r,s} n_q\,\mathrm{d}\Sigma,$$

where n_q is the unit outward normal to the surface $\overline{W}$, and $\mathrm{d}\Sigma$ the element of surface area.

Thus
$$\frac{\mathrm{d}E}{\mathrm{d}h} = -\frac{1}{2}\int I\,\mathrm{d}\Sigma, \tag{3·2}$$

where
$$I = \sigma(\dot{u}_p \dot{u}_p + c_{pqrs} u_{p,q} u_{r,s}) - 2c_{pqrs} u_p n_q u_{r,s}. \tag{3·3}$$

Completing the square and making use of the positivity of strain energy, we find

$$\begin{aligned}\sigma I &= \sigma^2 \dot{u}_p \dot{u}_p + c_{pqrs}(\sigma u_{p,q} - \dot{u}_p n_q)(\sigma u_{r,s} - \dot{u}_r n_s) - c_{pqrs} \dot{u}_p n_q \dot{u}_r n_s \\ &\geqslant \sigma^2 \dot{u}_p \dot{u}_p - c_{pqrs} \dot{u}_p n_q \dot{u}_r n_s. \end{aligned}\tag{3·4}$$

We now show that this last quantity is non-negative, where σ is the inward normal velocity of the surface $\overline{W}(x, t-h)$ as h increases. This velocity is given by the slope of the normal $(y_r, 1)$ relative to the hyperplane $h = 0$, and so is

$$\tan\theta = 1/|\mathbf{y}| = \bar{v}(\mathbf{y}),$$

since $(y_r, 1)$ lies on the inner slowness surface S_1. Now since $\bar{v}^2(y)$ is the largest characteristic root of $K_{pr}(y)$, we have

$$\sigma^2 = \bar{v}^2(\mathbf{y}) = \max_{|n|=1}(K_{pr}(\mathbf{y})\, n_p n_r) = \max_{|n|=1}(c_{pqrs} y_p n_q y_r n_s), \tag{3·5}$$

and it follows that the second term of the right side never exceeds the first one. Hence $I \geqslant 0$, and so $\mathrm{d}E/\mathrm{d}h \leqslant 0$. If at time zero we have $u_p = 0$, $\dot{u}_p = 0$, then $E(0) = 0$ and consequently $E(h) \equiv 0$ for $0 \leqslant h < t$. Thus $\dot{u}_p$ vanishes inside the retrograde convex wave cone with vertex at $(\mathbf{x}, t)$, and by continuity u_p is therefore zero at $(\mathbf{x}, t)$. This completes the proof that the value of $u_p(\mathbf{x}, t)$ depends only on the initial data within the convex envelope $\overline{W}(\mathbf{x}, t)$.

The quantity $\rho E(t)$ represents the total energy of the elastic vibrations, and we include a short proof, based on formula (3·2), of the conservation of energy. Suppose that at time zero the field is at rest except in a certain bounded domain D_0, where the initial displacement and velocity vectors are given smooth vector fields vanishing near the boundary of D_0. Then, at a later time t the disturbance is confined to a larger domain D_t which is bounded by the envelope of $\overline{W}(\mathbf{x}, t)$, when x traverses the boundary of D_0. Let us fix a time interval t_0, and choose D_{t_0} as the domain of integration for the integral $E(t)$, $0 \leqslant t < t_0$. As the u_p vanish on the boundary of D_{t_0}, we have $I = 0$ in (3·3) and so (3·2) yields $\mathrm{d}E(t)/\mathrm{d}t = 0$.

Hence
$$E(t) = E(0), \tag{3·6}$$

an equation which is clearly true for all later time. This shows that the total of kinetic and strain energy of the field remains constant.

The sheets of the wave cone $W(\mathbf{x}, t)$ divide the interior of the convex envelope $\overline{W}(\mathbf{x}, t)$ into certain regions, the number of which will depend upon the relationship of the three sheets and the number of cuspidal edges. Let the region containing the point $\mathbf{x}$ be denoted

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by $R_0(\mathbf{x}, t)$. Rather than label each of the other regions individually, we find it convenient to denote by $R_N(\mathbf{x}, t)$ $(N = 1, 2, 3)$ the entire region interior to $\overline{W}(\mathbf{x}, t)$ and exterior to $W_N(\mathbf{x}, t)$. Regions adjacent to cuspidal edges shall be counted as exterior to the sheet in question.

It is possible that the continuous wave may be absent from one or more of these regions, which are then known as *lacunae*. Petrowsky (1945) has given a deep topological criterion as a necessary and sufficient condition for a given region to be a lacuna. We will show by a direct evaluation of the solution that the innermost region R_0 is a lacuna, and hence, as described earlier, the continuous wave terminates with the last sharp wave. Thus there is no 'residual' or 'diffused' continuous wave. The total domain of dependence for $u_p(\mathbf{x}, t)$ is then the interior and boundary of $\overline{W}(\mathbf{x}, t)$, less $R_0(\mathbf{x}, t)$ (figure 2).

A well-known example of a lacuna is connected with the strong form of Huyghens's premise (Courant & Hilbert 1937, p. 395). The propagation of light is clean-cut, without residual waves, and so the interior of the light-cone in space-time is a lacuna. The existence of lacunae depends in large measure on the parity of the number of space dimensions.

4. Explicit calculation of the general solution

Applying the Fourier transform to (1·2), integrating by parts and rejecting integrals over the infinite sphere, we find

$$\frac{\partial^2 Fu_p(\mathbf{x}, t)}{\partial t^2} = -K_{pr}(\mathbf{x})\, Fu_r(\mathbf{x}, t), \tag{4·1}$$

where again
$$K_{pr}(\mathbf{x}) = K_{rp}(\mathbf{x}) = c_{pqrs}\, x_q x_s. \tag{4·2}$$

This is satisfied by
$$Fu_p(\mathbf{x}, t) = A_p(\mathbf{x}) \sin[v(\mathbf{x})\, t], \tag{4·3}$$

provided that $v(x)$ satisfies the determinantal condition

$$\det[v^2\delta_{pr} - K_{pr}(\mathbf{x})] = 0, \tag{4·4}$$

while the eigenvector $A_p(x)$ satisfies

$$(v^2\delta_{pr} - K_{pr}(\mathbf{x}))\, A_r(\mathbf{x}) = 0 \quad (p = 1, 2, 3). \tag{4·5}$$

The positive definite character of $K_{pr}(\mathbf{x})$ assures the existence of the three roots $v_N^2(x)$ described in § 2, with

$$v_1^2(x) \geqslant v_2^2(x) \geqslant v_3^2(x). \tag{4·6}$$

We assume henceforth that $v_N(\mathbf{x})$ $(N = 1, 2, 3)$ is positive, and it is easy to show that $v_N(\mathbf{x})$ is positively homogeneous of degree one in $\mathbf{x}$: $v_N(\lambda\mathbf{x}) = |\lambda|\, v_N(\mathbf{x})$ for any scalar λ. If $A_p^N(\mathbf{x})$ is an eigenvector corresponding to the eigenvalue $v_N^2(\mathbf{x})$, then any such eigenvector has the form $C^N(\mathbf{x})\, A_p^N(\mathbf{x})$, where N is *not* summed. Thus by superposing three expressions such as (4·3) we find the general solution

$$Fu_p(\mathbf{x}, t) = \sum_{N=1}^{3} C^N(\mathbf{x})\, A_p^N(\mathbf{x}) \sin[v_N(\mathbf{x})\, t]. \tag{4·7}$$

Here the summation over N is explicit, as N is not a tensorial index. Now (4·7) satisfies (4·2) and will satisfy the initial conditions if

$$\sum_{N=1}^{3} C^N(\mathbf{x})\, A_p^N(\mathbf{x})\, v_N(\mathbf{x}) = Fg_p(\mathbf{x}). \tag{4·8}$$

To determine the $C^N(\mathbf{x})$ from these conditions, we note that the vectors $A_p^N(\mathbf{x})$ are independent and so an inverse matrix $E_N^p(\mathbf{x})$ can be defined with the property that

$$E_M^p(\mathbf{x})\,A_p^N(\mathbf{x}) = \delta_M^N. \tag{4·9}$$

Then (4·8) yields
$$C^N(\mathbf{x}) = \frac{E_N^p(\mathbf{x})\,Fg_p(\mathbf{x})}{v_N(\mathbf{x})},$$

and (4·7) may be written

$$Fu_p(\mathbf{x},t) = \sum_{N=1}^{3} L_{pq}^N(\mathbf{x})\,Fg_q(\mathbf{x})\sin\,[v_N(\mathbf{x})\,t], \tag{4·10}$$

where
$$L_{pq}^N(\mathbf{x}) = \frac{A_p^N(\mathbf{x})\,E_N^q(\mathbf{x})}{v_N(\mathbf{x})}, \tag{4·11}$$

and N is not summed.

When two or more of the $v_N(\mathbf{x})$ coincide in value, there is a lack of uniqueness in the choice of the eigenvectors $A_p^N(\mathbf{x})$, but it is easy to verify that (4·10) is independent of the actual choice made.

Finally, therefore, the solution is

$$u_p(\mathbf{x},t) = \sum_{N=1}^{3} F^{-1}\{L_{pq}^N(\mathbf{x})\,Fg_q(\mathbf{x})\sin\,[v_N(\mathbf{x})\,t]\}, \tag{4·12}$$

and in the following section we study the reduction of this sixfold integral.

Explicit calculation of the $L_{pq}^N(x)$ is best attained by starting from Kelvin's concise form of the fundamental cubic equation for v^2. This is presented by Musgrave (1954*a*); we modify the notation slightly in order to use indices. Since $K_{pr}(x)$ is a symmetric 3×3 matrix, it is possible to write

$$K_{pr} = \alpha_p\alpha_r, \quad (p \neq r) \tag{4·13}$$

and to solve for the three quantities α_p:

$$\alpha_p^2 = \frac{K_{pr}K_{ps}}{K_{rs}},$$

where p, r, s are distinct. Thus α_p is homogeneous of degree $+1$ in the x_r. Setting

$$a_r = K_{rr} - \alpha_r^2,$$

we find that (4·5) can be written

$$(v^2 - a_p)\,A_p = \alpha_p\,\Sigma_r\,\alpha_r A_r = \alpha_p S,$$

say, where p is not summed on the left. Thus

$$A_p = \frac{\alpha_p S}{v^2 - a_p}, \tag{4·14}$$

and it is convenient to take $S \equiv \Sigma_r\,\alpha_r A_r = 1$. Thence follows Kelvin's form of the cubic equation, namely

$$\Sigma_r\frac{\alpha_r^2}{v^2 - a_r} = 1. \tag{4·15}$$

The eigenvector $A_p^N(\mathbf{x})$ can therefore be taken as

$$A_p^N(\mathbf{x}) = \frac{\alpha_p}{v_N^2 - a_p}. \tag{4·16}$$

In calculating the inverse matrix, let us adopt the convention that $p, p+1, p+2$ shall denote the three index values 1, 2, 3 in some order. Then, after some calculation and reduction, we find

$$E_N^p(x) = \alpha_p \frac{(v_N^2 - a_{p+1})(v_N^2 - a_{p+2})}{(v_N^2 - v_{N+1}^2)(v_N^2 - v_{N+2}^2)}, \tag{4·17}$$

and so

$$L_{pq}^N(x) = \frac{\alpha_p \alpha_q}{v_N(v_N^2 - a_p)} \frac{(v_N^2 - a_{q+1})(v_N^2 - a_{q+2})}{(v_N^2 - v_{N+1}^2)(v_N^2 - v_{N+2}^2)}. \tag{4·18}$$

It will be noted that in the regular case the three terms in (4·10), which correspond to $N = 1, 2, 3$, are well defined separately and independently, since the three roots $v_N^2(x)$ are all distinct. In the next section (§ 5) we restrict ourselves to this regular case, and study the reduction of the multiple integrals.

The singular case offers two difficulties: first, the singular points on the slowness surface which make precautions necessary in defining the integrals; and, secondly, the behaviour of the factors $L_{pq}^N(x)$. In § 6 it will be shown that a singular problem is, in a certain sense, the limit of a regular one, and the calculations for the regular case will be taken over by a limiting process.

5. Reduction of the integrals, regular case

We now select one of the three terms in (4·12), and omit the index N for convenience. Thus we consider

$$u_p(\mathbf{x}, t) = F^{-1}\{L_{pq} F g_q \sin[v(\mathbf{x})\, t]\}, \tag{5·1}$$

where $L_{pq}(x)$ is positively homogeneous of degree -1 in x. Then

$$(2\pi)^3\, u_p(x, t) = \iint e^{i\mathbf{y}(\mathbf{z}-\mathbf{x})} \sin[v(\mathbf{y})\, t]\, L_{pq}(\mathbf{y})\, g_q(\mathbf{z})\, d\mathbf{y}\, d\mathbf{z} \tag{5·2}$$

and by noting the evenness in y of $v(y)$ and $L_{pq}(y)$, and replacing $\mathbf{z}-\mathbf{x}$ by $\mathbf{z}$, we find

$$(2\pi)^3\, u_p(\mathbf{x}, t) = \iint \cos[\mathbf{y}.\mathbf{z}] \sin[v(\mathbf{y})\, t]\, L_{pq}(\mathbf{y})\, g_q(\mathbf{x}+\mathbf{z})\, d\mathbf{y}\, d\mathbf{z}. \tag{5·3}$$

We now proceed to reduce this sextuple integral, a rather delicate matter. If we introduce polar co-ordinates by writing

$$\mathbf{y} = |\mathbf{y}|\, \boldsymbol{\eta} \quad (|\boldsymbol{\eta}| = 1),$$
$$\mathbf{z} = |\mathbf{z}|\, \boldsymbol{\zeta} \quad (|\boldsymbol{\zeta}| = 1),$$

we have

$$d\mathbf{y} = |\mathbf{y}|^2\, d|\mathbf{y}|\, d\omega_\eta,$$
$$d\mathbf{z} = |\mathbf{z}|^2\, d|\mathbf{z}|\, d\omega_\zeta,$$

where $d\omega_\eta$, $d\omega_\zeta$ are elementary solid angles. We then have integrations over $|\mathbf{y}|$ and $|\mathbf{z}|$ from 0 to ∞, and over the two unit spheres. Care must be taken to avoid divergent integrals. The plan is as follows:

(1) integrate with respect to $|\mathbf{y}|$ from 0 to R;
(2) integrate with respect to $d\omega_\eta$;
(3) apply the limit $R \to \infty$, conjointly with integration over $|\mathbf{z}|$, and use Dirichlet's formula (1·6);
(4) integrate with respect to $d\omega_\zeta$.

Let t be positive, and suppose $g_p(x)$ vanishes outside a bounded domain D, the region of initial disturbance. Write (5·3) in the form

$$2(2\pi)^3 u_p(\mathbf{x}, t) = \lim_{R\to\infty} \int g_q(\mathbf{x}+\mathbf{z})\,\mathrm{d}\mathbf{z} \int L_{pq}(\eta)\, I_R(\mathbf{z}, t, \eta)\,\mathrm{d}\omega_\eta, \tag{5·4}$$

where
$$I_R(\mathbf{z}, t, \eta) = 2\int_0^R \cos\,[|\mathbf{y}|\boldsymbol{\eta}\,.\,\mathbf{z}]\sin\,[|\mathbf{y}|\,v(\eta)\,t]\,|\mathbf{y}|\,\mathrm{d}|\mathbf{y}|. \tag{5·5}$$

This last integral is elementary: if we set

$$\xi = \boldsymbol{\eta}\,.\,\mathbf{z} - v(\eta)\,t; \quad \xi' = \boldsymbol{\eta}\,.\,\mathbf{z} + v(\eta)\,t, \tag{5·6}$$

the integral becomes

$$I_R(\mathbf{z}, t, \eta) = \frac{R\xi\cos R\xi - \sin R\xi}{\xi^2} - \frac{R\xi'\cos R\xi' - \sin R\xi'}{\xi'^2}$$
$$= \frac{\mathrm{d}}{\mathrm{d}\xi}\left(\frac{\sin R\xi}{\xi}\right) - \frac{\mathrm{d}}{\mathrm{d}\xi'}\left(\frac{\sin R\xi'}{\xi'}\right). \tag{5·7}$$

Noting that the second part of (5·7) is an odd function of ξ', we can show that upon integration with respect to $\mathrm{d}\omega_\eta$ the contribution from the second term duplicates that from the first. A reversal of sign of η beneath the integral sign leads at once to this conclusion.

Thus
$$(2\pi)^3 u_p(\mathbf{x}, t) = \lim_{R\to\infty} \int g_q(\mathbf{z}+\mathbf{x})\,\mathrm{d}\mathbf{z} \int L_{pq}(\eta)\,\frac{\mathrm{d}}{\mathrm{d}\xi}\left(\frac{\sin R\xi}{\xi}\right)\mathrm{d}\omega_\eta. \tag{5·8}$$

The form of the inner integral is well adapted to the use of Dirichlet's formula (1·6), provided that first we integrate by parts with respect to ξ. We therefore first study the behaviour of ξ on the unit η-sphere. It will be seen below that it is actually better to use arc length on the unit η-sphere as variable, instead of ξ.

For given z, t the quantity ξ will attain one or more maximum and minimum values on the unit η-sphere. Let $\xi^{(2)}$ denote such a local maximum, and $\xi^{(1)}$ the absolute minimum value. Since $t > 0$, and $v(\eta) > 0$, it follows that $\xi^{(1)} < 0$. Let $\eta^{(1)}$ and $\eta^{(2)}$ be the corresponding points of the η-sphere, found from the conditions

$$\delta\xi = z_r\delta\eta_r - t\frac{\partial v}{\partial\eta_r}\delta\eta_r = \theta\eta_r\delta\eta_r,$$

so that
$$z_r - t\frac{\partial v}{\partial \eta_r} = \theta\eta_r, \tag{5·9}$$

with $\eta_r\eta_r = 1$. These four equations will determine η_r and θ as functions of z and t. We note that
$$\theta = \eta_r z_r - tv(\eta) = \xi, \tag{5·10}$$

at the stationary points $\eta^{(1)}$ and $\eta^{(2)}$.

We now join the two points $\eta^{(1)}$ and $\eta^{(2)}$ by arcs of circles on the sphere, each such arc being specified by an azimuthal angle ϕ. Let s be arc length, measured from $\eta^{(1)}$, and let the normal separation of the arcs be $n(s, \phi)\,\mathrm{d}\phi$. Since $\mathrm{d}\omega_\eta = n(s, \phi)\,\mathrm{d}s\,\mathrm{d}\phi$ we write

$$J_R(z, t) = \int L_{pq}(\eta)\,\frac{\mathrm{d}}{\mathrm{d}\xi}\left(\frac{\sin R\xi}{\xi}\right)\mathrm{d}\omega_\eta$$
$$= \int_0^{2\pi}\mathrm{d}\phi\int_{\eta^{(1)}}^{\eta^{(2)}} L_{pq}(\eta)\,\frac{\mathrm{d}}{\mathrm{d}s}\left(\frac{\sin R\xi}{\xi}\right)\frac{\mathrm{d}s}{\mathrm{d}\xi}\,n(s, \phi)\,\mathrm{d}s$$
$$= \int_0^{2\pi}\mathrm{d}\phi\left[L_{pq}(\eta)\,\frac{\sin R\xi}{\xi}\,\frac{\mathrm{d}s}{\mathrm{d}\xi}\,n(s, \phi)\right]_{\eta^{(1)}}^{\eta^{(2)}}$$
$$-\int_0^{2\pi}\mathrm{d}\phi\int_{\eta^{(1)}}^{\eta^{(2)}}\frac{\sin R\xi}{\xi}\,\frac{\mathrm{d}}{\mathrm{d}s}\left[L_{pq}(\eta)\,\frac{\mathrm{d}s}{\mathrm{d}\xi}\,n(s, \phi)\right]\mathrm{d}s. \tag{5·11}$$

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The integration by parts has now been performed, with respect to the variable s. At this point the separation of the solution into sharp and continuous waves is effected, the integrated term leading to the sharp wave.

In order to rewrite the sharp wave portion in a more convenient form, we note that the integrated terms arise from the two critical points $\eta^{(1)}$ and $\eta^{(2)}$. As we approach $\eta^{(1)}$, where $d\xi/ds = 0$ by the minimal property of ξ, we have $n(s, \phi) \sim s$, and

$$\frac{d\xi}{ds} \sim s\left(\frac{d^2\xi}{ds^2}\right)_{\eta^{(1)}}.$$

Therefore
$$\frac{ds}{d\xi} n(s,\phi) = 1 \Big/ \frac{d^2\xi}{ds^2} \quad \text{at} \quad \eta^{(1)}. \tag{5·12}$$

Likewise, it is found that at $\eta^{(2)}$:

$$\frac{ds}{d\xi} n(s,\phi) = -1 \Big/ \frac{d^2\xi}{ds^2}. \tag{5·13}$$

Accordingly we have from (5·8), (5·11), (5·12) and (5·13),

$$(2\pi)^3 u_p(\mathbf{x}, t) = \lim_{R\to\infty} \int g_q(\mathbf{z}+\mathbf{x})\, d\mathbf{z} [K^R_{pq}(\mathbf{z}, t) + M^R_{pq}(\mathbf{z}, t)], \tag{5·14}$$

where the kernel of the sharp wave is

$$K^R_{pq}(\mathbf{z}, t) = L_{pq}(\eta^{(2)}) \frac{\sin R\xi^{(2)}}{\xi^{(2)}} \int_0^{2\pi} d\phi \Big/ \left(\frac{d^2\xi}{ds^2}\right)_{\eta^{(2)}} + L_{pq}(\eta^{(1)}) \frac{\sin R\xi^{(1)}}{\xi^{(1)}} \int_0^{2\pi} d\phi \Big/ \left(\frac{d^2\xi}{ds^2}\right)_{\eta^{(1)}}, \tag{5·15}$$

and the kernel for the continuous wave is

$$M^R_{pq}(\mathbf{z}, t) = \int_0^{2\pi} d\phi \int_{\eta^{(1)}}^{\eta^{(2)}} \frac{\sin R\xi}{\xi} \frac{d}{d\xi}\left[L_{pq}(\eta) \frac{ds}{d\xi} n(s,\phi)\right] d\xi. \tag{5·16}$$

(a) *The sharp wave*

We now consider the contribution from the first term of $K^R_{pq}(\mathbf{z}, t)$ as $R \to \infty$. Because of the oscillating factor $\sin[R\xi^{(2)}]$, the limit of the contribution will be zero unless $\xi^{(2)}$ vanishes within the domain of integration. Since

$$\xi^{(2)} = |z|\, \boldsymbol{\eta}^{(2)}.\boldsymbol{\zeta} - tv(\eta^{(2)}),$$

this will happen when this expression has opposite signs for $|z| = r_1$, and $|z| = r_2$, the limits of integration over D in the direction $\boldsymbol{\zeta}$. Since $\xi^{(1)} < 0$, the other term contributes nothing.

Let us consider the meaning of the condition $\xi^{(2)} = 0$. We have

$$\xi^{(2)} = \eta^{(2)}_r . z_r - v(\eta^{(2)})\, t = \theta^{(2)} = 0,$$

and so, by (5·9)
$$z_r = t\left(\frac{\partial v}{\partial \eta_r}\right)_{\eta^{(2)}}. \tag{5·17}$$

Referring to (2·4) we see that this states just that the point z_r shall lie upon the wave surface $W(t)$, which is the unit wave surface W expanded in the ratio of t to 1. However, (5·17) also demands that $\boldsymbol{\eta}$ and $\boldsymbol{\zeta}$ should be conjugate unit vectors with respect to the slowness and wave surfaces, in the following sense. Let the ray defined by $\boldsymbol{\eta}$ cut the slowness surface at a point A. Then $\boldsymbol{\zeta}$ has the direction of the normal at A to the slowness surface, and so is uniquely determined. Since the relation between the slowness and wave surfaces is reciprocal, the normal at a point of W, cut by the ray $\boldsymbol{\zeta}$ from the origin, has the direction $\boldsymbol{\eta}$.

However, as a sheet of W may be cut by such a ray in more than one point (if cusp-lines are present), the conjugate $\boldsymbol{\eta}(\boldsymbol{\zeta})$ is multiple-valued in general. The appropriate value can be determined by continuity, if necessary.

We now choose a point of observation $\mathbf{x}$ and a time t, and draw the observer's wave surface $W(x, t)$ with centre $\mathbf{x}$. Let $\boldsymbol{\zeta}$ define a ray drawn from $\mathbf{x}$; then the sole contribution from $K^R_{pq}(\mathbf{x}, t)$ comes from that $|\mathbf{z}|$ where the ray cuts $W(\mathbf{x}, t)$. In order to express the limit as $R \to \infty$ as an integral over $W(t)$, we note that the cosine of the angle between the ray from the origin and the normal to W is $\boldsymbol{\eta}.\boldsymbol{\zeta}$, where $\boldsymbol{\eta}$ is conjugate to $\boldsymbol{\zeta}$ as above. Thus

$$\mathrm{d}\mathbf{z} = |\mathbf{z}|^2\,\mathrm{d}|\mathbf{z}|\,\mathrm{d}\omega_\zeta = \eta.\zeta\,\mathrm{d}|\mathbf{z}|\,\mathrm{d}S_W, \tag{5·18}$$

where $\mathrm{d}S_W$ is the surface element of W. By Dirichlet's formula (1·6) we find from (5·14) and (5·15) that the sharp wave contribution is

$$\pi\int_{W(t)} g_q(r\boldsymbol{\zeta}+\mathbf{x})\,\eta.\zeta\,\mathrm{d}S_W\left(\frac{\mathrm{d}|\mathbf{z}|}{\mathrm{d}\xi^{(2)}}\right)_{\xi^{(2)}=0} = L_{pq}(\eta^{(2)})\left[\int_0^{2\pi}\mathrm{d}\phi\Big/\left(\frac{\mathrm{d}^2\xi}{\mathrm{d}s^2}\right)_{\eta^{(2)}}\right]_{\xi^{(2)}=0}. \tag{5·19}$$

Here $\mathrm{d}S_W$ is the surface element of W in the z variables, and $\boldsymbol{\eta}^{(2)} = \boldsymbol{\eta}^{(2)}(\boldsymbol{\zeta})$ is the appropriate value of the direction $\boldsymbol{\eta}$ conjugate to $\boldsymbol{\zeta}$. Also r is the radius vector from $\mathbf{x}$ to $W(\mathbf{x}, t)$ in the $\boldsymbol{\zeta}$ direction.

To evaluate the integral in the square bracket in (5·19), we note that

$$\frac{\mathrm{d}\xi}{\mathrm{d}s} = z_r\frac{\mathrm{d}\eta_r}{\mathrm{d}s} - t\frac{\mathrm{d}v}{\mathrm{d}\eta_r}\frac{\mathrm{d}\eta_r}{\mathrm{d}s}$$

and
$$\frac{\mathrm{d}^2\xi}{\mathrm{d}s^2} = \left(z_r - t\frac{\partial v}{\partial\eta_r}\right)\frac{\mathrm{d}^2\eta_r}{\mathrm{d}s^2} - t\frac{\partial^2 v}{\partial\eta_r\,\partial\eta_s}\frac{\partial\eta_r}{\partial s}\frac{\mathrm{d}\eta_s}{\mathrm{d}s}. \tag{5·20}$$

Since the integral in (5·19) is to be evaluated when $\xi^{(2)} = 0$, the first term on the right of (5·20) must vanish, by (5·17). Thus the integral is equal to $-\psi(\eta^{(2)})/t$, where

$$\psi(\eta^{(2)}) = \left[\int_0^{2\pi}\frac{\mathrm{d}\phi}{\dfrac{\partial^2 v}{\partial\eta_r\,\partial\eta_s}\dfrac{\mathrm{d}\eta_r}{\mathrm{d}s}\dfrac{\mathrm{d}\eta_s}{\mathrm{d}s}}\right]_{\eta^{(2)}(\zeta)}. \tag{5·21}$$

This integral may be evaluated by taking the maximum point as north pole of the η-sphere in spherical polar co-ordinates, and by use of the formula

$$\int_0^{\pi}\frac{\mathrm{d}\theta}{a+\cos\theta} = \frac{\pi}{\surd(a^2-1)} \quad (a>1).$$

The result is
$$\psi(\eta^{(2)}) = \frac{\pm 2\pi}{\surd\{|K(\eta^{(2)})|\}}\quad \boldsymbol{\eta}^{(2)}.\boldsymbol{\zeta}, \tag{5·22}$$

where $K(\eta)$ is the Gaussian curvature of the slowness surface (Eisenhart 1909, p. 123). We note that K is positive if the slowness surface is strictly convex, in which case the positive sign should be taken in (5·22). A parabolic line on the slowness surface will lead to a singularity of $\psi(\eta)$; we will later show that this singularity is integrable. For hyperbolic or concave regions, if any, of the slowness surface, the negative sign shall be taken in (5·22).

We must also consider the ratio
$$(\mathrm{d}|\mathbf{z}|/\mathrm{d}\xi^{(2)})_{\xi^{(2)}=0},$$
where $\xi^{(2)} = |\mathbf{z}|\,\boldsymbol{\eta}^{(2)}.\boldsymbol{\zeta} - v(\eta^{(2)})\,t$, and

$$|\mathbf{z}|\,\zeta_r - t(\partial v/\partial\eta_r)_{\eta^{(2)}} = \xi^{(2)}\eta_r^{(2)}, \tag{5·23}$$

with $\eta_r\eta_r = 1$. Elimination of the $\eta_r^{(2)}$ will yield $|\mathbf{z}|$ as a function of $\xi^{(2)}$; to find the above derivative we vary the independent quantities in (5·23):

$$\zeta_r \delta |z| - t\frac{\partial^2 v}{\partial\eta_r \partial\eta_s}\delta\eta_s - \eta_r \delta\xi^{(2)} - \xi^{(2)}\delta\eta_r = 0. \tag{5·24}$$

Now eliminate the differentials of the η_r by means of the second equation of variation $\eta_r\delta\eta_r = 0$. To do this, multiply (5·24) by η_r and contract over r. Since $\partial v/\partial\eta_s$ is homogeneous of degree zero in the η_r, the term with the second derivatives in (5·24) will vanish, by Euler's theorem. The result becomes

$$\left(\frac{d|\mathbf{z}|}{d\xi^{(2)}}\right)_{\xi^{(2)}=0} = \frac{1}{\boldsymbol{\eta}^{(2)}.\boldsymbol{\zeta}}. \tag{5·25}$$

Substituting these results in (5·19), and simplifying, we find the general formula for the sharp wave portion of the solution contributed by the Nth sheet, namely,

$$v_p^N(\mathbf{x},t) = \frac{1}{4\pi t}\int_{W_N(t)} L_{pq}^N(\eta)\, g_q(r\zeta+\mathbf{x})\,\frac{\boldsymbol{\eta}.\boldsymbol{\zeta}\, dS_W}{\epsilon\sqrt{\{|K(\eta)|\}}}. \tag{5·26}$$

Here $\boldsymbol{\eta}$ and $\boldsymbol{\zeta}$ are conjugate as before, dS_W is the element of surface area of $W(t)$, in the variables $\boldsymbol{\zeta}$, and the sign of $\epsilon = \epsilon_N(\eta)$ is to be taken as the sign of the Gaussian curvature $K(\eta)$.

By means of the geometric relations between the slowness and wave surfaces we can put the above integral into more convenient forms. Since $\boldsymbol{\eta}.\boldsymbol{\zeta}$ is the cosine of the inclination of the normal of $W(t)$ to the radius vector from the origin, we have

$$\boldsymbol{\eta}.\boldsymbol{\zeta}\, dS_W = r_0^2 t^2\, d\omega_\zeta, \tag{5·27}$$

where $r = r_0 t$ is the distance $|\mathbf{z}|$. In order to relate $d\omega_\zeta$ to the area element dS of the slowness surface, let us use the original definition of Gauss for the curvature $K(\eta)$: it is the reciprocal of the ratio of dS to the area element on the 'hodograph' or spherical indicatrix related to this surface: and this sphere is exactly the ζ-sphere. Thus (Eisenhart 1909, p. 145)

$$d\omega_\zeta = K(\eta)\, dS. \tag{5·28}$$

Since $W = W(1)$ and S are reciprocal surfaces, there are relations dual to (5·27) and (5·28): they are

$$\boldsymbol{\eta}.\boldsymbol{\zeta}\, dS = |\mathbf{y}|^2\, d\omega_\eta, \tag{5·29}$$

and

$$t^2\, d\omega_\eta = K_W(\zeta)\, dS_W. \tag{5·30}$$

Here $K_W(\zeta)$ is the Gaussian curvature of the wave surface $W(1)$: the curvature for $W(t)$ is proportional to t^{-2} which accounts for the factor on the left of (5·30).

Multiply together these four relations: we find

$$(\eta.\zeta)^2 = r_0^2\,|\mathbf{y}|^2\, K(\eta)\, K_W(\zeta). \tag{5·31}$$

This also shows that the two dual Gauss curvatures have the same sign; and therefore (5·31) is true, when the curvatures are replaced by their absolute values. As in (2·3) we have $|\mathbf{y}|\, v(\eta) = 1$, and if we take square roots in (5·31) we have for the sharp wave the expressions

$$\begin{aligned} v_p^N(\mathbf{x},t) &= \frac{1}{4\pi t}\int_{W_N(t)} L_{pq}^N(\eta)\, g_q(r\boldsymbol{\zeta}+\mathbf{x})\,\frac{r_0(\zeta)}{v_N(\eta)}\,.\,\epsilon\sqrt{\{|K_W(\zeta)|\}}\, dS_W \\ &= \frac{t}{4\pi}\int_{W_N(1)} L_{pq}^N(\eta)\, g_q(r_0 t\boldsymbol{\zeta}+\mathbf{x})\frac{r_0(\zeta)}{v_N(\eta)}\,.\,\epsilon\sqrt{\{|K_W(\zeta)|\}}\, dS_W. \end{aligned} \tag{5·32}$$

The sharp wave term may be transformed into an integral over the slowness surface, a form which, though less suited to physical interpretation would, however, be easier to evaluate numerically. By (5·27) and (5·28) we find

$$v_p^N(\mathbf{x},t) = \frac{t}{4\pi}\int_{S_N} L_{pq}^N(\eta)\, g_q(r_0 t\boldsymbol{\zeta}+\mathbf{x})\, r_0^2 \, . \, \epsilon \sqrt{\{|K(\eta)|\}}\, \mathrm{d}S. \tag{5·33}$$

Here the radius vector $\mathbf{z} = r_0 t\boldsymbol{\zeta}$ in the argument of g_q can be expressed as a function of $\boldsymbol{\eta}$ by the parametric formulas given by Musgrave (1954*a*, p. 349).

The curvature $K(\eta)$ vanishes on the inflexion lines of S, which correspond to the cusps of W. Formula (5·33) shows that the contribution to the integral near these singularities is bounded.

(*b*) *The continuous wave*

We have now to consider the contribution from $M_{pq}^R(\mathbf{z},t)$ in (5·14). Unless ξ vanishes in the range of integration, we have

$$\lim_{R\to\infty} M_{pq}^R(\mathbf{z},t) = 0,$$

as follows easily from (1·6); and thus zero contribution. We can therefore speak of $\mathbf{z}$, t contributing to the continuous wave if

$$\xi^{(2)} = \max_{|\eta|=1} (\mathbf{z}\,.\,\boldsymbol{\eta} - v(\eta)\,t)$$

is positive; and we seek the contributing region as follows. Let $\mathbf{z} = |\mathbf{z}|\,\boldsymbol{\zeta}$, and note that for $\xi^{(2)} = 0$ the point $\mathbf{z}$ lies on the wave cone $W(t)$, by (5·17). Since $v(\eta)\,t > 0$, we must have $\mathbf{z}\,.\,\boldsymbol{\eta} = |\mathbf{z}|\,\boldsymbol{\zeta}\,.\,\boldsymbol{\eta} > 0$ and it follows that if $\mathbf{z}$ lies outside the wave cone $W(t)$ there are positive values of ξ for $\boldsymbol{\eta} = \boldsymbol{\eta}^{(2)}(\zeta)$. On the other hand, if $\mathbf{z}$ lies within $W(t)$ the maximum of ξ is clearly negative. We conclude that the domain of contributing $\mathbf{z}$ is the region exterior to the wave cone $W(t)$; that is, exterior to the particular sheet $W_N(t)$ now under consideration.

From (5·16) we see that if z contributes, then by (1·6)

$$M_{pq}(\mathbf{z},t) = \lim_{R\to\infty} M_{pq}^R(\mathbf{z},t) = \pi\int_0^{2\pi} \mathrm{d}\phi\left[\frac{\mathrm{d}s}{\mathrm{d}\xi}\frac{\mathrm{d}}{\mathrm{d}s}\left(L_{pq}(\eta)\frac{\mathrm{d}s}{\mathrm{d}\xi}n(s,\phi)\right)\right]_{(\xi=0)}. \tag{5·34}$$

Let us define

$$F_{pq}(\bar\xi) = \int_{\xi=\eta\cdot\mathbf{z}-v(\eta)t<\bar\xi} L_{pq}(\eta)\,\mathrm{d}\omega_\eta$$

$$= \int_{\xi<\bar\xi} L_{pq}(\eta)\, n(s,\phi)\frac{\mathrm{d}s}{\mathrm{d}\xi}\,\mathrm{d}\phi\,\mathrm{d}\xi. \tag{5·35}$$

Then

$$\frac{\mathrm{d}F_{pq}(\bar\xi)}{\mathrm{d}\bar\xi} = \oint_{\xi=\bar\xi} L_{pq}(\eta)\, n(s,\phi)\frac{\mathrm{d}s}{\mathrm{d}\xi}\,\mathrm{d}\phi$$

and

$$\frac{\mathrm{d}^2F_{pq}(\bar\xi)}{\mathrm{d}\bar\xi^2} = \oint \mathrm{d}\phi\left[\frac{\mathrm{d}}{\mathrm{d}\xi}\left(L_{pq}(\eta)\, n(s,\phi)\frac{\mathrm{d}s}{\mathrm{d}\xi}\right)\right]_{\xi=\bar\xi}. \tag{5·36}$$

Hence

$$M_{pq}(\mathbf{z},t) = \pi\left[\frac{\mathrm{d}^2F_{pq}(\bar\xi)}{\mathrm{d}\bar\xi^2}\right]_{\bar\xi=0}. \tag{5·37}$$

This expression is the kernel of the integral for the continuous wave. Indeed from (5·14) we see that the contribution to the continuous wave from the particular value N considered is

$$w_p^N(\mathbf{x},t) = \frac{1}{8\pi^2}\int g_q(\mathbf{z}+\mathbf{x})\,\mathrm{d}\mathbf{z}\left[\frac{\mathrm{d}^2}{\mathrm{d}\bar\xi^2}\int_{\xi_N<\bar\xi} L_{pq}^N(\eta)\,\mathrm{d}\omega_\eta\right]_{\bar\xi=0}. \tag{5·38}$$

Here the integration over $\mathbf{z}$ is taken over the contributing region, namely that part of the domain D exterior to any part of the sheet $W_N(t)$ of the wave surface $W(t)$. The function $M_{pq}(\mathbf{z}, t)$ vanishes, however, in the region interior to the wave surface, since then all values of $\boldsymbol{\eta}\,.\,\mathbf{z}-v(\eta)\,t$ are negative and the integration is carried out over the entire η-sphere. Thus $F_{pq}(\xi)$ is a constant for all non-negative ξ and so $M_{pq}(\mathbf{z}, t)$ vanishes.

This establishes the fact, mentioned in § 3, that the innermost region R_0 of the three-sheeted wave cone is a lacuna. Indeed there is no contribution for any value $N = 1, 2, 3$ from this region.

The domain of integration in (5·38) can be bounded on the outside if we recall that the entire domain of dependence, that is, the entire domain of integration for the integrals appearing in the solution, is contained in the convex closure $\overline{W}(\mathbf{x}, t)$ of the observer's wave cone. Since the argument of the datum functions $g_p(\mathbf{x})$ contains the translation vector x, we may take the domain of integration in (5·38) as limited outside by $\overline{W}(t)$. Thus the regions $R_N(t)$ defined in § 3 as exterior to $W_N(t)$ and interior to $\overline{W}(t)$ shall be chosen as the proper domains of integration for the integrals (5·38). We note that if $W_1(t)$ is convex then $R_1(t)$ is empty and in that case only two of the continuous wave terms (5·38) will appear. This cancellation of integrals over the infinite domain was noted by Stokes in his solution for the isotropic case, which is examined in § 7 (*a*) below. It is an expression of the fact that wave propagation with arbitrarily large velocity cannot take place.

The full expression for the solution is

$$u_p(\mathbf{x}, t) = \frac{1}{4\pi t}\sum_{N=1}^{3}\int_{W_N(t)} L_{pq}^N(\eta)\, g_q(r_0\boldsymbol{\zeta}+\mathbf{x})\,\frac{r_0^N(\zeta)}{v_N(\eta)}\,\epsilon_N(\eta)\,\surd\{|K_W^N(\zeta)|\}\,\mathrm{d}S_W^N$$
$$+\frac{1}{8\pi^2}\sum_{N=1}^{3}\int_{R_N(t)} g_q(\mathbf{z}+\mathbf{x})\,\mathrm{d}\mathbf{z}\left[\frac{\mathrm{d}^2}{\mathrm{d}\bar{\xi}^2}\int_{\xi_N<\bar{\xi}} L_{pq}^N(\eta)\,\mathrm{d}\omega_\eta\right]_{\bar{\xi}=0}. \tag{5·39}$$

Here $W_N(t)$ is the Nth sheet of the wave surface $W(t)$, the cross-section at time t of the wave cone, and $R_N(t)$ is the region enclosed between $W_N(t)$ and the convex envelope $\overline{W}(t)$ of the 3-sheeted wave surface $W(t)$. Here also ξ_N denotes the combination $\boldsymbol{\eta}\,.\,\mathbf{z}-v_N(\eta)t$.

From this expression we see that an initial disturbance concentrated near the origin of physical space causes a continuous (or volume) wave and a sharp (or shell-like) wave to spread between and along the sheets of the direct wave cone W with vertex at the origin. At a field point x the continuous wave will in general begin first, although if $W_1(t)$ is convex the first sharp wave will arrive simultaneously with the continuous wave. The continuous wave is punctuated by subsequent sharp waves and terminates at the instant of the last sharp wave, which is attached to the innermost sheet of the wave surface. As stated earlier the number of sharp waves is half the degree of $W(t)$; that is, half the class number of the slowness surface, and therefore cannot exceed the maximum of 75. In special cases this number can be greatly reduced.

6. Multiple points on S

That certain types of multiple points do necessarily appear on S, in crystal classes of physical importance, has been made evident by the work of Musgrave. We need to show that the solution formulae of § 5 have a meaning in such cases. The known examples of physical interest involve only double points, and we shall give a complete description of

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the solution for these. Our analysis does not cover all possible types of triple points, as there is one circumstance (Case VI below) wherein the factors L^N_{pq} have a singularity.

The components u_p all satisfy the single sixth order equation $Lu_p = 0$, where

$$L_k = S\left(\frac{\partial}{\partial t}, \frac{\partial}{\partial x}\right) = \det\left[\delta_{pr}\frac{\partial^2}{\partial t^2} - c_{pqrs}\frac{\partial^2}{\partial x^q\,\partial x^s}\right]; \qquad (6\cdot1)$$

this equation has the same slowness and wave surfaces as the system. If now there are multiple points on S, we shall perturb the sixth order operator L in such a way as to separate the multiple roots, while ensuring that the perturbed operator L_ϵ is regularly hyperbolic. With $S(t,y)$ defined as in (2·2), we shall write

$$S_k(t,y) = \frac{\partial^k}{\partial(t^2)^k} S(t,y) \quad (k = 1, 2); \qquad (6\cdot2)$$

$S_k(t,y)$ is of degree $3-k$ in t^2, and of degree $6-2k$ in y. If now the slowness surface $S(1,y) = 0$ has a triple point, the surface $S_1(1,y) = 0$ has a double point coincident with it, by Rolle's theorem. Indeed, the sheets of the cone $tS_1(t,y) = 0$ separate the sheets of $S(t,y) = 0$; and a similar relationship holds between tS_2 and S_1. It follows that the two sheets of the surface $S_1(1,y) = 0$ in y-space separate the three sheets of $S(1,y) = 0$, and are in turn separated by the one sheet of $S_2(1,y) = 0$. Examination of the regions in which these expressions are positive or negative shows that the surface $S(1,y) - \epsilon_1 S_1(1,y) = 0$, for $\epsilon_1 > 0$, has a double point; the third sheet being detached from it. This surface also has three sheets. A second perturbation will resolve the double points: thus the surface

$$S(1,y) - (\epsilon_1 + \epsilon_2)\, S_1(1,y) + \epsilon_1\epsilon_2 S_2(1,y) = 0$$

has three sheets and is free of multiple points.

We now consider the differential operator

$$L_\epsilon \equiv S\left(\frac{\partial}{\partial t}, \frac{\partial}{\partial x}\right) - (\epsilon_1+\epsilon_2)\frac{\partial^2}{\partial t^2} S_1\left(\frac{\partial}{\partial t}, \frac{\partial}{\partial x}\right) + \epsilon_1\epsilon_2\frac{\partial^4}{\partial t^4} S_2\left(\frac{\partial}{\partial t}, \frac{\partial}{\partial x}\right), \qquad (6\cdot3)$$

which is homogeneous of the sixth order with constant coefficients. Clearly its slowness surface is free of multiple points: if we denote the three velocities of propagation by v_N we have $v_1^\epsilon > v_2^\epsilon > v_3^\epsilon$. By a calculation analogous to that of § 4 we find that the solution of this equation, with initial values corresponding to the conditions $u_p = 0$; $\partial u_p/\partial t = g_p$ for $t = 0$, takes the form (4·12), where, however, the $v_N(x)$ and $L^N_{pq}(x)$ now depend upon ϵ_1 and ϵ_2. Details of this calculation are omitted.

We now sketch a proof that, for a given set of sufficiently smooth initial data, the solutions u_p depend continuously upon ϵ_1 and ϵ_2. Suppose for simplicity that $\epsilon_1 = \epsilon_2 = \epsilon$. Then

$$u_p(\epsilon) = u_p(0) + \epsilon\,\partial u_p(\theta\epsilon)/\partial\epsilon \quad (0 < \theta < 1)$$

by Taylor's theorem. Now if $v_p(\theta\epsilon) \equiv \partial u_p(\theta\epsilon)/\partial\epsilon$, then v_p satisfies an inhomogeneous equation

$$L_{\theta\epsilon} v_p = 2\frac{\partial^2}{\partial t^2} S_1\left(\frac{\partial}{\partial t}, \frac{\partial}{\partial x}\right) u_p(\theta\epsilon) - 2\theta\epsilon\frac{\partial^4}{\partial t^4} S_2\left(\frac{\partial}{\partial t}, \frac{\partial}{\partial x}\right) u_p(\theta\epsilon),$$

which is clearly of the regularly hyperbolic type, i.e. its slowness surface has three non-intersecting sheets. The solution of such an equation and its derivatives will satisfy energy integral estimates (Leray 1953) which can in this case be established uniformly with respect to ϵ as $\epsilon \to 0$. It follows from Sobolev's lemma (Leray, p. 159) that v_p satisfies a similar pointwise bound. This shows that u_p depends continuously upon ϵ as $\epsilon \to 0$, and therefore that the solution in the case of multiple points is the limit of the solution of $L_\epsilon u_p = 0$ with corresponding initial data.

We have therefore to perform the limit $\epsilon \to 0$ in (5·39). It is helpful to regard the sharp wave terms as constituting one integral extended over the complete surface $W(t)$. The integrand is to be defined, at ordinary points, by specifying the root which gives the corresponding portion of the slowness surface. Formally, the sharp wave is

$$\lim_{\epsilon \to 0} \int_{W_\epsilon(t)} L^\epsilon_{pq}(\eta) g_q(r_0 t\boldsymbol{\zeta} + \mathbf{x}) \frac{r_0(\zeta)}{v(\eta)} \epsilon(K) \sqrt{\{K_W(\zeta)\}}\, dS_W.$$

We consider double points on S, and note that the limit of $W_\epsilon(t)$ is $\overline{W}(t)$, the convex completion of $W(t)$. The formal limit is therefore

$$\int_{\overline{W}(t)} L_{pq}(\eta)\, g_q(r_0 t\boldsymbol{\zeta} + \mathbf{x}) \frac{r_0(\zeta)}{v(\eta)} \epsilon(K) \sqrt{\{K_W(\zeta)\}}\, dS_W. \tag{6·4}$$

We assert, however, that this is the same as the integral over $W(t)$.

To indicate that convergence difficulties are absent, we shall show that over those parts of $\overline{W}$ which correspond to singularities of S, the integrand is actually zero. We recall from § 2 that to a conical double point of S corresponds a planar portion of W, which has zero Gaussian curvature. This is a particular case of the fact that the reciprocal of a surface having singularities is defined as the limit of the reciprocals to neighbouring 'perturbed' surfaces on which the singularities have been reduced. To any singular component of S which is of dimension 1 or 0 will correspond a portion of W which is a ruled surface or a portion of a plane, respectively. Thus $K_W(\zeta) = 0$ in both cases, which suggests that it is sufficient to extend the integration over $W(t)$.

To complete the proof we shall show that $L^N_{pq}(\eta)$ is bounded on S. In describing the behaviour of $L_{pq}(\eta)$ we shall use the Kelvin form of the fundamental cubic equation for $v^2(\eta)$. We must examine the circumstances which permit the factors in the denominator of the expression

$$L^N_{pq}(\eta) = \frac{\alpha_p \alpha_q}{v_N^2 - a_p} \frac{1}{v_N} \frac{(v_N^2 - a_{q+1})(v_N^2 - a_{q+2})}{(v_N^2 - v_{N+1}^2)(v_N^2 - v_{N+1}^2)}, \tag{6·5}$$

to vanish—that is, when two roots coincide or when one or more roots equal one of the a_p.

Now the three roots v_N^2 are given by the abscissae of the intersections of the quartic curve

$$y = \sum \frac{\alpha_p^2}{x - a_p} \tag{6·6}$$

with the line $y = 1$. We shall study the quartic curve, which in general has three vertical asymptotes $x = a_p$ ($p = 1, 2, 3$), since if $\alpha_p \neq 0$, the function $y(x)$ has an infinity as $x \to a_p$. The three quantities a_p separate the three roots v_N^2 as is evident from figure 6; there are two cases according as the α_p^2 are all positive or all negative, and the figure shows the positive case.

Case I. One root $v^2(\eta)$ is equal to an a_p. This can happen if the corresponding α_p is zero. Then if $p = q$, the leading factors in L^N_{pq} become

$$\frac{\alpha_p^2}{v_N^2 - a_p} = 1 - \sum_{q \neq p} \frac{\alpha_q^2}{v_N^2 - a_q},$$

by the fundamental cubic. The right side is bounded, provided only that the a_p are distinct. For $p \neq q$ the factor $v_N^2 - a_p$ in the denominator will cancel with one of the factors $v_N^2 - a_{q+i}$ ($i = 1$ or 2), in the numerator.

One root can be equal to an a_p if two of the a_p coincide: then one branch of the quartic is the line $x = a_p$. Again there will be cancellation of the factor $v^2 - a_p$ in the denominator with one of the similar factors in the numerator.

Case II. Two roots v^2 coincide. This can happen if all three a_p coincide, in which case two roots equal their common value. The corresponding L_{pq}^N have at most two factors vanishing in the denominator, and there are two vanishing factors in the numerator. (All of these factors vanish to the first order when a parameter ϵ and a perturbed surface $S_0 + \epsilon S_1 = 0$ are introduced.) The third L_{pq}^N ($N = 3$) has a non-zero denominator unless all three roots coincide.

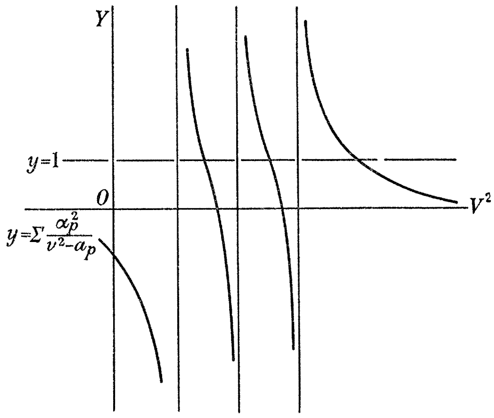

FIGURE 6. The quartic curve, intersection of which with the line $y = 1$ gives the roots v_N^2 of the fundamental cubic. The abscissae of the three vertical asymptotes are the a_p ($p = 1, 2, 3$). The case where $\alpha_p^2 > 0$ ($p = 1, 2, 3$) is sketched; in the contrary case the curve resembles the reflexion in the V^2 axis of the example shown.

We consider this case of 3 equal roots, when the a_p are all equal. After multiplying up and reducing, we find the fundamental cubic is

$$(v^2 - a)^2 (v^2 - a - \alpha_1^2 - \alpha_2^2 - \alpha_3^2) = 0.$$

Since the α_p are either all real or all pure imaginary, we can have $\alpha_1^2 + \alpha_2^2 + \alpha_3^2 = 0$ only if $\alpha_1 = \alpha_2 = \alpha_3 = 0$. As the product $\alpha_p \alpha_q = K_{pq}$ (for $p \neq q$) will vanish to the first order, both numerator and denominator of L_{pq}^N vanish to the third order, with a finite limiting ratio.

Case III. Two a_p equal, with one α_p vanishing. Suppose $a_1 = a_2$. If α_1 or α_2 vanishes, it can be seen from the graph of the quartic curve that there will be no double root, and we are back to case I. Suppose then $\alpha_3 = 0$, and that $\alpha_1 \neq 0$, $\alpha_2 \neq 0$. This will happen only if $K_{13} = K_{23} = 0$ but $K_{12} \neq 0$; and moreover α_3 vanishes to the first order. One root is $v_3^2 = a_3$, and the residual quadratic equation is

$$\frac{\alpha_1^2}{v^2 - a_1} + \frac{\alpha_2^2}{v^2 - a_2} = 1,$$

so that if a_1 and a_2 are equal, the other two roots are $v_1^2 = a_1$ and $v_2^2 = a_1 + \alpha_1^2 + \alpha_2^2$. If now $N = p = 1$, the factor $(v_1^2 - a_1)$ in the denominator vanishes, but one of the factors $(v_1^2 - a_{q+1})$ for $i = 1$ or 2 will cancel it off. For $N = p = 3$, the ratio $\alpha_p/(v_3^2 - a_3)$ is finite.

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Another difficulty appears if $a_1+\alpha_1^2+\alpha_2^2 = a_3$ in the limit: then $v_2 = v_3$, and for $N = 2$ and 3 there is a vanishing factor in the denominator. However, there is also a factor vanishing to the first order in the numerator; for $q = 3$ this is α_q and for $q \neq 3$ it is that one of the factors $v_N^2-a_{q+1}$ which contains a_3. Hence L_{pq}^N is bounded in the limit.

Case IV. One α_p vanishes; the a_p are unequal. Suppose $\alpha_3 = 0$; then $v_3^2 = a_3$ is a root, and the quotient $\alpha_3/(v_3^2-a_3)$ remains bounded in the limit as in Case III. Neither a_1 or a_2 is a root of the residual quadratic equation. If one of the roots v_1^2 or v_2^2 of this equation is equal to a_3, there is a vanishing factor in the denominator of L_{pq}^N for two values of N. However it is cancelled as in Case III, either by α_q $(q = 3)$ or by $v_N^2-a_{q+i}$ $(q \neq 3,\ q+i = 3)$.

Case V. Two α_p vanish. Let $\alpha_1 = \alpha_2 = 0$; two roots are $v_1^2 = a_1$, $v_2^2 = a_2$ and the third root is $v_3^2 = a_3+\alpha_3^2$, which for the moment will be assumed distinct from the others. If $v_N^2-a_p$ in the denominator is zero, we may proceed as in Case I. Equal roots can arise if $a_1 = a_2$, in which case another factor in the denominator will vanish. If both $v_N^2-a_p$ and $v_N^2-v_{N+1}^2$ vanish, which is possible for $p = 1$ or 2, we have to find two factors vanishing to the first order in the numerator. If $q = 1$ or 2, the product $\alpha_p\alpha_q$ provides one of these factors. The other is one of the factors $v_N^2-a_{q+i}$ (for $q+i \neq 3$). However, if $q = 3$ both factors of this latter type will vanish, as $a_1 = a_2$ is the common value of the double root.

Case VI. Three roots equal. From the graph of the quartic equation it is seen that for all three roots to coincide, two of the α_p, say α_1 and α_2 must vanish, while the corresponding a_p are equal to the common root. The third value for v^2 is $a_3+\alpha_3^2$ which may be equal to a_1 and a_2. If α_3 is not zero then $a_3 \neq a_1 = a_2$, and L_{pp}^N will have a singularity, as three factors in the denominator but only two in the numerator will vanish, for instance if $p = q = N = 1$. This case is therefore not covered in our general remarks on the convergence of the integral (6·4).

To conclude, we see that $L_{pq}^N(\eta)$ is a bounded factor in the integrand, in the presence of double points. As the integral (6·4) is the correct sharp wave for a perturbed system, and as $K_W(\zeta) \to 0$ near multiple points of S, it follows that (6·4) gives the solution in these degenerate cases also, the integration not being extended over the ruled surfaces or planar domains corresponding to multiple lines or points of S. When such singularities are present on the innermost sheet of S, there will exist certain directions in which the continuous wave reaches the field point in advance of the first sharp wave.

7. Particular examples

(a) *The isotropic case*

As an illustration of the quantities appearing in this solution, and also as a numerical check, the formulae have been worked out for the isotropic case, for which there are only two independent elastic constants. If we denote by λ and μ these two constants divided by the density, then the slowness surface consists of a sphere of multiplicity one corresponding to the pressure wave velocity

$$v_1 = \sqrt{(\lambda+2\mu)}\,|\mathbf{x}|,$$

and a sphere of multiplicity two corresponding to the twofold shear wave velocity:

$$v_2 = v_3 = \sqrt{\mu}\,|\mathbf{x}|.$$

The matrix $K_{pr}(x)$ has the form $\mu\,|\mathbf{x}|^2\,\delta_{pr}+(\lambda+\mu)\,x_p x_r$, and, in the notation of § 4, we have $\alpha_p = \sqrt{(\lambda+\mu)}\,x_p$, while $a_p = \mu\,|\mathbf{x}|^2$. The A_p^N are not unique, and are best chosen to satisfy equations (4·5) directly. This can be done by taking A_p^N as one or other of the co-ordinates x_r, with a minus sign in some instances. Calculating the $L_{pq}^N(\mathbf{x})$ directly, we find

$$L_{pq}^1 = \frac{x_p x_q}{\surd(\lambda+2\mu)\,|\mathbf{x}|^3}; \quad L_{pq}^2 + L_{pq}^3 = \frac{|\mathbf{x}|^2\,\delta_{pq} - x_p x_q}{\surd\mu\,|\mathbf{x}|^3}.$$

The kernels $M_{pq}^N(z,t)$ can be constructed first for $z = (z,0,0)$, and then for general positions of z, using the Cartesian tensor property of the indices p and q. For this it is necessary to evaluate only two easy elementary integrals. The result is

$$M_{pq}^1(\mathbf{z},t) = -\frac{2\pi^2 t}{|\mathbf{z}|^3}\left(3\frac{z_p z_q}{|\mathbf{z}|^2} - \delta_{pq}\right) \quad (v_1 t < |\mathbf{z}|),$$
$$= 0 \quad (v_1 t \geqslant |\mathbf{z}|);$$

while
$$M_{pq}^2(\mathbf{z},t) + M_{pq}^3(\mathbf{z},t) = \frac{2\pi^2 t}{|\mathbf{z}|^3}\left(3\frac{z_p z_q}{|\mathbf{z}|^2} - \delta_{pq}\right) \quad (v_2 t < |\mathbf{z}|),$$
$$= 0 \quad (v_2 t \geqslant |\mathbf{z}|).$$

The cancellation outside the wave surface is evident.

The explicit solution of the initial value problem is

$$u_p(\mathbf{x},t) = \frac{t}{4\pi}\int_{\omega_\zeta} \zeta_p \zeta_q g_q(\surd(\lambda+2\mu)\,t\zeta + \mathbf{x})\,\mathrm{d}\omega_\zeta$$
$$+\frac{t}{4\pi}\int_{\omega_\zeta} (\delta_{pq} - \zeta_p\zeta_q)\,g_q(\surd\mu\,t\zeta + \mathbf{x})\,\mathrm{d}\omega_\zeta$$
$$+\frac{t}{4\pi}\iint_{\surd\mu\,t<|\mathbf{z}|<\surd(\lambda+2\mu)t} (3\zeta_p\zeta_q - \delta_{pq})\,g_q(\mathbf{z}+\mathbf{x})\,\mathrm{d}\omega_\zeta \frac{\mathrm{d}|\mathbf{z}|}{|\mathbf{z}|},$$

where the double integral represents the continuous wave, and $\mathbf{z} = |\mathbf{z}|\,\zeta$, $|\zeta| = 1$. This formula agrees with Stokes's solution (1882, p. 268).

(*b*) *The cubic case*

In this example we make use of the Kelvin form of the fundamental cubic for v^2 to determine the multiple points or curves on the slowness surface. This enables one to give a closer estimate of the class number of S and hence of the maximum number of sharp waves which will spread out in any direction from a source. The cubic slowness and wave surfaces have been studied in Miller & Musgrave (1956) and we shall use the same notation for the elastic constants of the medium.

The Kelvin form of the cubic for v^2 is

$$\sum_{r=1}^{3} \frac{b x_r^2}{v^2 - c_{44}\,|\mathbf{x}|^2 + (b-a)\,x_r^2} = 1,$$

where a, b, and c_{44} are the three independent constants present. Now unless one or more of the numerators in the three terms of this equation vanishes, the three quantities

$$a_r = c_{44}\,|\mathbf{x}|^2 - (b-a)\,x_r^2$$

are separated by two of the roots v^2. Thus if two of these quantities are equal, there is a root v^2 equal to both, and if all three agree, two equal roots v^2 are found having the common value. The third root for v^2 is larger than all of these three expressions if b is positive, and smaller if b is negative. Thus the only double points not on the co-ordinate planes are on the planes bisecting the angles between them. There are necessarily eight double points, on the lines

$\pm x_1 = \pm x_2 = \pm x_3$. These are conical points and it has been noted (Miller & Musgrave 1956) that they give rise to internal conical refraction.

If $b \neq 0$, the numerators $\alpha_p^2 = bx_p^2$ vanish only on the co-ordinate planes, and the cubic equation factors there. The two outer sheets of S touch at their points of intersection with the co-ordinate axes, and so there are six double points of the type known as unodes (Sommerville 1934, p. 375).

As a conical point subtracts 2 and a unode 6 from the class, we have

$$\text{class} \leqslant 150 - 8 \times 2 - 6 \times 6 = 98,$$

so that the number of sharp waves cannot exceed 49. For copper, as plotted by Miller & Musgrave (1956), there appear to be 9 sharp waves in directions close to the co-ordinate axes.

For $b > 0$ the inner sheet S_1 does not cross over S_2 and S_3 and so is convex. The dual outer sheet W_1 of the wave surface is also convex, and so the convex envelope coincides with it. Therefore the continuous wave begins simultaneously with the first sharp wave (figure 3).

For $b = 0$, the slowness surface degenerates into three congruent oblate spheroids, each having a different co-ordinate axis as axis of revolution. There are then eight triple points in place of the above conical points. The wave surface, consisting of three prolate spheroids, one having each axis as major axis, is not uniformly convex (figure 4). The convex envelope $\overline{W}$ is formed by adjoining eight portions of plane surfaces and twelve portions of cylindrical surfaces to the outer sides of these spheroids. These correspond to the eight triple points and twelve segments of double curves of intersection of the three oblate spheroids of the slowness surface. Near the eight directions in physical space which are equally inclined to the crystal axes, the continuous wave front will precede the sharp waves.

For $b < 0$ the eight conical points lie on the two innermost sheets of S. Consequently the two outer algebraic sheets of W form a not uniformly convex surface, and the convex envelope $\overline{W}$ must be constructed as in the case $b = 0$.

(c) *The hexagonal case*

An axis of rotational symmetry is present, and as shown by Musgrave (1954*b*) this allows the construction of the wave surface as a surface of revolution based on the curve reciprocal to a section of the slowness surface. There are five independent elastic constants. The cubic equation for v^2 factors and the cross-section of the slowness surface by a plane through the axis of revolution is a conic C together with a quartic curve consisting of two concentric non-intersecting closed curves Q_1 and Q_2.

As the class of the plane curve reciprocal to a curve of degree n is $\leqslant n(n-1)$, the class of the slowness surface is not more than $2 + 4 \times 3$, these being the contributions of the two parts. Consequently the greatest number of sharp waves possible in a medium of hexagonal symmetry is seven. An example where five are present is given by Musgrave. The conic, being an ellipse, is convex, and the inner sheet Q_1 of the quartic is also convex. Thus only the outer sheet Q_2 of the quartic has points of inflexion giving rise to cusps on W and hence multiple sharp waves.

The branches C and Q_2 touch as they cross the axis of revolution, and may have four further points of intersection, the condition for this being given by Musgrave. It can be shown that the energy condition (1·3) implies that the combination $a-\frac{1}{2}c$ (in Musgrave's notation) is positive, and it follows that C does not meet Q_1.

Since Q_1 is isolated, the innermost sheet S_1 of S, formed by revolving Q, is isolated and convex. Therefore the outer sheet W, which is the algebraic dual of S_1, is convex. That is, $W(x,t)$ coincides with its convex envelope $\overline{W}(x,t)$ and the continuous wave begins simultaneously with the first sharp wave.

8. A criterion for convexity of W_1

As the slowness surface has degree six, a straight line meeting all three sheets meets each sheet in just two points. If then there exists an innermost sheet S_1 isolated from S_2 and S_3, then S_1 is convex because it is met by no straight line in more than two points. Hence W_1 is convex and coincides with the convex envelope.

The Kelvin form (4·15) of the fundamental cubic may be used to show that this situation prevails in certain cases. The three quantities $\alpha_1^2(x)$, $\alpha_2^2(x)$, $\alpha_3^2(x)$ defined by (4·13) are all positive or all negative according as $K_{12}(x)\,K_{23}(x)\,K_{31}(x)$ is positive or negative. Supposing they are positive, we plot the left-hand side of (4·15) as a function of v^2 (figure 6) and observe that the greatest root $v_1^2(x)$ is the only root which exceeds all three quantities $a_p(x)$.

The condition $K_{12}(x)\,K_{23}(x)\,K_{31}(x) > 0$ thus ensures that there is no double point on the inner sheet S_1, in the x direction. For orthorhombic symmetry with nine independent constants (Musgrave 1957), this product is

$$(c_{1122}+c_{1212})\,(c_{2233}+c_{2323})\,(c_{3311}+c_{3131})\,x_1^2x_2^2x_3^2,$$

and if the product of constants is supposed positive, only the co-ordinate planes must be considered separately. On a co-ordinate plane the cubic factors and the three roots are found by solving a linear and a quadratic equation.

If now the innermost branch of each co-ordinate plane section is isolated, and the above constant is positive, then the sheet S_1 is isolated. It follows that S_1 and hence W_1 are convex.

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References

Courant, R. & Hilbert, D. 1931 *Methoden der Mathematischen Physik*, vol. 1. Berlin: Springer.
Courant, R. & Hilbert, D. 1937 *Methoden der Mathematischen Physik*, vol. 2. Berlin: Springer.
Eisenhart, L. P. 1909 *Differential geometry*. Boston: Ginn and Co.
Leray, J. 1953 *Hyperbolic differential equations*. Princeton, N.J.: Institute for Advanced Study.
Love, A. E. H. 1944 *The mathematical theory of elasticity*, 4th ed. New York: Dover.
Miller, G. F. & Musgrave, M. J. P. 1956 *Proc. Roy. Soc.* A, **236**, 352.
Musgrave, M. J. P. 1954*a* *Proc. Roy. Soc.* A, **226**, 339.
Musgrave, M. J. P. 1954*b* *Proc. Roy. Soc.* A, **226**, 356.
Musgrave, M. J. P. 1957 *Proc. Camb. Phil. Soc.* **53**, 897.
Petrowsky, I. G. 1945 *Mat. Sbornik*, **59** (n.s. 17), 289.
Salmon, G. 1882 *Geometry of three dimensions*. Dublin: Hodges, Foster and Figgis.
Sommerville, D. M. Y. 1934 *Analytical geometry of three dimensions*. Cambridge.
Stokes, G. G. 1883 *Mathematical and physical papers*, vol. 2. Cambridge University Press.
Synge, J. L. 1956 *Proc. R. Irish Acad.* A, **58**, 13.
Synge, J. L. 1957 *J. Math. Phys.* **35**, 323.
Wooster, W. A. 1938 *A text-book on crystal physics*. Cambridge University Press.

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